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On Critical Phenomena in Interacting Growth Systems. Part II: Bounded Growth

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This paper completes the classification of some infinite and finite growth systems which was started in Part I. Components whose states are integer numbers interact in a local deterministic way, in addition to which every component's state grows by a positive integer k with a probability $\varepsilon^k(1-\varepsilon)$ at every moment of the discrete time. Proposition 1 says that in the infinite system which starts from the state "all zeros," percentages of elements whose states exceed a given value $k \geqslant 0$ never exceed $(C\varepsilon)^k$, where $C = \text{const}$. Proposition 2 refers to finite systems. It states that the same inequalities hold during a time which depends exponentially on the system size.

KEY WORDS: Random process; local interaction; critical phenomena; growth; combinatorics; contour method; graph theory.

1. INTRODUCTION

This Part II completes the classification of some infinite and finite growth systems which was started in Part I.[3] Part I contains all the necessary definitions and notations as well as relevant concepts, examples, and references. Propositions 1 and 2 of this paper imply Theorems 1 and 2 of Part I.

Remember that we are dealing with an infinite or finite system of interacting elements, whose states are integer numbers. At every moment of the discrete time the elements interact in a local deterministic way, after which every elements's state is independently incremented by a random variable ζ. Throughout this paper

$$\text{Prob}(\zeta \geqslant k) = \varepsilon^k \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

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Of all the "standard assumptions" of Part I, we need only

$$\forall x_1,\dots, x_n: \quad \mathbf{f}(x_1,\dots, x_n) \leqslant \max\{x_1,\dots, x_n\}$$

Proposition 1. Take such a transition function and neighbor vectors that $\sigma = \{\mathcal{O}\}$. There is such a constant C that for all inner points p, all positive q, and all ε the infinite system satisfies the following:

$$\operatorname{Prob}(x(p) \geqslant q) \leqslant \begin{cases} \varepsilon + (C\varepsilon)^2 & \text{if } q = 1 \\ (C\varepsilon)^{q+1} & \text{if } q > 1 \end{cases} \tag{1}$$

Proposition 2. Take such a transition function and neighbor vectors that $\sigma = \{\mathcal{O}\}$. For any T there is such a constant C that the inequality (1) holds for all finite systems of size M, all inner points p, for which $t(p) \leqslant T^M$, all positive q, and all ε.

Sections 2–4 prove Proposition 1, the last section proves Proposition 2.

2. FAMILY $\Phi(p, q)$

As is typical of the contour method, we prove Proposition 1 by covering the event $x(p) \geqslant q$ by such a finite family $\Phi(p, q)$ of events called *patches* that

$$\sum_{P \in \Phi(p,q)} \operatorname{Prob}(P) \leqslant \begin{cases} \varepsilon + (C\varepsilon)^2 & \text{if } q = 1 \\ (C\varepsilon)^{q+1} & \text{if } q > 1 \end{cases} \tag{2}$$

where C is a positive constant. A *patch* is a subset of the hidden configuration space, specified by the condition $h(v) \geqslant P(v)$ for all iner v, where P is a map $P\colon V_{\text{inner}} \mapsto \{0, 1, 2, 3,\dots\}$. A map P of this sort given, we designate the corresponding patch by the same letter P. The set $\operatorname{dom}(P) = \{v \mid P(v) > 0\}$ is called the *domain* of the patch P. Actually we consider only those patches whose probability is positive, that is, those whose domains are finite. The whole hidden configuration space is a special patch whose domain is empty. Note that the probability of any patch P equals $\varepsilon^{\operatorname{sum}(P)}$, where $\operatorname{sum}(P)$, or the *sum* of a patch P, denotes the sum of $P(\cdot)$ over its domain. We call "inquest" the process of construction patches, in the course of which we trace the events $x(p) \geqslant q$ back to its original causes—*upstrats*. An upstart is an inner point v, where $x(v) = h(v) > 0$. The set of upstarts is denoted $U(h)$.

2.1. The Linear Lemma

Lemma 1. Condition $\sigma = \{\mathcal{O}\}$ is equivalent to the following: There are such a natural number $r \leqslant s+1$ and such r homogeneous linear functions $L'_1, ..., L'_r$ on R^{d+1} that

$$L'_1 + \cdots + L'_r \equiv 0 \tag{3}$$

and

$$\text{for all } i = 1, ..., r \text{ the set } \{e \in N(\mathcal{O}) \mid L'_i(e) \leqslant -1\} \text{ is an } \mathcal{O}\text{-drag} \tag{4}$$

Proof. First assume $\sigma = \{\mathcal{O}\}$. Take the functions $L_1, ..., L_r$ provided by the Linear Lemma in ref. 3 and define $L'_i = r \cdot L_i + t$, which evidently fit our claims. Proof in the other direction is similar to the first part of the proof of the Linear Lemma in ref. 3. ∎

From now on we choose the smallest possible r and some homogeneous linear functions $L'_1, ..., L'_r$ which satisfy (3) and (4).

2.2. Polars and Trusses

In the classical contour method a contour consists of directed elements which can be imagined as small "magnets," every one of which has a positive and a negative pole. When these elements make a contour, all the poles are "neutralized," which means that each pole of every element coincides with the opposite pole of another element. Our constructions come to these when $r = 2$. In the general case we use branching analogs of countours, which we call trusses (as in ref. 1). They are made of analogs of "magnets," whtch we call polars. Polars may have poles of r different kinds. Now we use the adjective "even" rather than "neuralized" to denote a similar, but more general property.

Let *vector field* ϕ on a set S denote a map ϕ: $S \to \mathbf{Z}^r$. A vector fiels ϕ given, $\phi(v)$ is its value on any $v \in S$; this value is a vector whose components are $\phi(v)_1, ..., \phi(v)_r$. For any $Q \subseteq S$ denote $\phi(Q) = \sum_{v \in Q} \phi(v)$. Call a vector field ϕ *even* or *p-even* on a set Q if all the components of $\phi(Q)$ equal p. Call a vector field on S *overall-even* if it is even on all subsets of S. A vector field ϕ on a set S given, its *domain* is $\mathrm{dom}(\phi) = \{v \in S \mid \phi(v) \neq 0\}$. Call a vector field *trivial* if its domain consists of one element. There is the zero vector field, which equals 0 on every element and has an empty domain.

Definition. Call a nonzero vector field π on S a *polar* if all of the following conditions hold:

- For any $v \in S$ and any $j = 1, \ldots, r$ the value of $\pi(v)_j$ is either 0 or 1 or -1.
- For every $j = 1, \ldots, r$ there is at most one point v where $\pi(v)_j = 1$; if it exists, it is called ther jth *positive pole* of π and denoted $\pi^+(j)$.
- For every $j = 1, \ldots, r$ there is at most one point v where $\pi(v)_j = -1$; if it exists, it is called the jth *negative pole* of π and denoted $\pi^-(j)$.
- π is even on S.

Note that any polar has at least two and at most $2 \cdot r$ poles (which may coincide). Any polar belongs to one of the following classes:

1. A *posipole*—a polar which has all the r positive poles and none negative.
2. A *negapole*—a polar which has all the r negative poles and more positive.
3. A polar which has both j-poles for all $j \in S$ and no other poles, where S is some subset of $\{1, \ldots, r\}$.

Call a *dipole* or a *j-dipole* a polar of the last class for which S consists of one element j. (Dipoles are the only polars of the last class we shall actually use.) We say that a dipole is directed from its positive pole to its negative pole. Thus a j-dipole π from u to v has $\pi^+(j) = u$ and $\pi^-(j) = v$ (and no other poles). (A dipole is like a "magnet" we mentioned before. Indeed, in the case $r = 2$ all the polars we need are dipoles.) Le us call two sequences of polars equivalent if one turns into the other by some permutation, and call the resulting classes of equivalence *trusses.* $\pi \in \mathscr{T}$ means that the polar π is a member of the truss $\mathscr{T}$, but different members of a truss may coincide. $|\mathscr{T}|$ denotes the number of members in the truss $\mathscr{T}$. We designate a truss by any sequence $\mathscr{T} = \langle \pi_1, \ldots, \pi_k \rangle$ of its members. j-poles of members of a truss are called its j-poles and $\mathscr{T}(j)$ denotes the set of j-poles of the truss $\mathscr{T}$. For any two trusses $\mathscr{A}$ and $\mathscr{B}$ their concatenation $\mathscr{A} * \mathscr{B}$ results from writing one sequence after the other. Thus $\mathscr{T} * \langle \pi \rangle$ means the truss $\mathscr{T}$ to which polar π is added. Conversely, $\pi \in \mathscr{T}$ given, $\mathscr{T} - \pi$ means $\mathscr{T}$ from which π is excluded. Of course,

$$|\mathscr{A} * \mathscr{B}| = |\mathscr{A}| + |\mathscr{B}|, \qquad |\mathscr{T} * \langle \pi \rangle| = |\mathscr{T}| + 1, \qquad |\mathscr{T} - \pi| = |\mathscr{T}| - 1$$

For any truss $\mathscr{T}$ its domain $\mathrm{dom}(\mathscr{T})$ is defined as the union of domain of its members. There is the empty truss, whose number of members is 0 and whose domain is empty. Call a truss $\mathscr{T}$ *connected* if for any $a, b \in \mathrm{dom}(\mathscr{T})$ there is a sequence $c_0 = a$, $c_1, \ldots, c_l = b$ in which every two neighboring terms belong to the domain of some member of $\mathscr{T}$.

To any truss $\mathscr{T}$ there corresponds its vector field vec($\mathscr{T}$), which is the sum of its members as vector fields. A truss $\mathscr{T}$ given, vec($\mathscr{T}$, v) and vec($\mathscr{T}$, S) denote the values of the corresponding vector field on the point v and set S. Call a truss p-even, even, or overall-even if its vectors field is.

For any r-tuple $F=(F_1,\ldots,F_r)$ for functions on a set S, any polar π and truss $\mathscr{T}$ on S, and any $v\in S$ denote

$$\operatorname{sum}(F,\pi,v)=\sum_{j=1}^{r} F_j(v)\cdot\pi(v)_j$$

$$\operatorname{sum}(F,\pi)=\sum_{v\in S}\operatorname{sum}(F,\pi,v)$$

$$\operatorname{sum}(F,\mathscr{T})=\sum_{\pi\in\mathscr{T}}\operatorname{sum}(F,\pi)$$

2.3. Graphs

We denote all graphs by letters with bars. They have no loops and every two vertices a and b are connected with at most one edge, which is denoted $a-b$ if it is nondirected, and $a\to b$ if it is directed from a to b. If $\bar{H}$ is a subgraph of $\bar{G}$, we write $\bar{H}\subseteq\bar{G}$. A graph $\bar{G}$ given, ver($\bar{G}$) denotes the set of its vertices.

Given any directed graph $\bar{G}$, call those vertices, whence edges go to a vertex, *$\bar{G}$-neighbors* of this vertex. The set of $\bar{G}$-neighbors of a vertex v is called in *$\bar{G}$-neighborhood* and denoted $N_{\bar{G}}(v)$. The $\bar{G}$-neighborhood of a set S of vertices is the union of $\bar{G}$-neighborhoods of elements of S. Further, $N^k_{\bar{G}}(S)$ is defined for every $k=0,1,2,\ldots$ and every $S\subseteq\operatorname{ver}(\bar{G})$ by the inductive rule:

$$N^0_{\bar{G}}(S)=S,\qquad N^{k+1}_{\bar{G}}(S)=N_{\bar{G}}(N^k_{\bar{G}}(S))$$

The *transit* and *proper* $\bar{G}$-neighborhoods of S are

$$N^{\text{tran}}_{\bar{G}}(S)=\bigcup_{k=0}^{\infty}N^k_{\bar{G}}(S)\qquad\text{and}\qquad N^{\text{prop}}_{\bar{G}}(S)=\bigcup_{k=1}^{\infty}N^k_{\bar{G}}(S)$$

and their elements are called transit and proper $\bar{G}$-neighbors of S, respectively. Call two vertices *$\bar{G}$-comparable* if one is a transit $\bar{G}$-neighbor of the other. Call two sets $\bar{G}$-comparable if some element of the first and some element of the second are $\bar{G}$-comparable; otherwise these sets are $\bar{G}$-uncomparable.

A polar on a graph $\bar{G}$ means a polar on ver($\bar{G}$). A polar π on a graph is called an *edger* on this graph if dom(π) consists of two vertices connected with an edge. We say that π lies on this edge and that this edge underlies π. A posiedger is an edger which is a posipole. A *diedger* on a directed graph is a dipole from u to v where u and v are vertices of our graph which contains an edge from u to v.

To represent the system of neighborhoods, we shall use the directed graph $\bar{V}$ which has V as its set of vertices, and edges that go to every point from its neighbors. Thus neighborhoods $N(\cdot)$ without indices, which we used in ref. 3, to define our systems, and $\bar{V}$-neighborhoods now, We shall also use the nondirected graph $\bar{V}'$ with V as its set of vertices, in which two vertices are connected if one is the other's neighbor or they are different neighbors of one point. A j-diedger on $\bar{V}$ from u to v is called an *arrow* or a *j-arrow* from u to v if $L_j'(u) - L_j'(v) \leqslant -1$.

Lemma 2. Given an overall-even truss $\mathscr{A} * \mathscr{E}$ on V, where $\mathscr{A}$ consists of arrows and $\mathscr{E}$ consists of edgers on $\bar{V}'$, then $|\mathscr{A}| \leqslant |\mathscr{E}| \cdot \text{const}$.

Proof. Note that if a truss $\mathscr{T}$ on V is overall-even, then sum $(L', \mathscr{T}) = 0$, where $L' = (L_1', ..., L_r')$. [This can be proved by grouping together addends that pertain to one and the same point and using (3).] Also note that $\text{sum}(L', \alpha) \geqslant 1$ for any arrow α and $|\text{sum}(L, \varepsilon)| \leqslant \text{const}$ for any edger ε on $\bar{V}'$. [One may take this $\text{const} = 2 \cdot \Delta \cdot \Lambda'$, where norm and Δ were defined in Part I and Λ' is the maximum of $L_i'(v)$ for $\text{norm}(v) \leqslant 1$ and $i = 1, ..., r$.] Now

$$0 = \text{sum}(L', \mathscr{A} * \mathscr{E}) = \text{sum}(L', \mathscr{A}) + \text{sum}(L', \mathscr{E}) \geqslant |\mathscr{A}| - \text{const} \cdot |\mathscr{E}| \quad \blacksquare$$

The following construction will be used whenever our "inquest" will ramify. Given a posipole π on a connected nondirected graph $\bar{G}$, its (nonunique) *spanning kit* consists of a *spanning tree* span-tree(π, $\bar{G}$) and a *spanning truss* span-truss(π, $\bar{G}$), which are defined as follows: First, span-tree(π, $\bar{G}$) is a minimal connected subgraph of $\bar{G}$ whose set of vertices contains dom(π). Of course, span-tree(π, $\bar{G}$) is a tree, whence every edge of it is a bridge. Based on this, for every edge $a - b$ of span-tree(π, $\bar{G}$) we form a posiedger ε_{a-b} whose kth pole for any $k = 1, ..., r$ coincides with a or b, namely with that one of the two which does not remain connected with $\pi(k)$ if the edge $a - b$ is deleted from span-tree(π, $\bar{G}$). Now span-truss(π, $\bar{G}$) consists of these posiedgers for all edges of span-tree(π, $\bar{G}$). We shall refer to the following properties of any spanning kit:

$$|\text{ver(pan-tree}(\pi, \bar{G}))| = |\text{span-truss}(\pi, \bar{G})| + 1 \tag{5}$$

(because any tree's number of vertices is one more than the number of its edges), truss$\langle\pi\rangle *$ span-truss$(\pi, \bar{G})$ is 1-even on every element of its domain (6)

and for any truss $\mathscr{T}$

if $\mathscr{T} * \langle\pi\rangle$ is connected, then $\mathscr{T} *$ span-truss$(\pi, \bar{G})$ is also connected (7)

2.4. Description of $\Phi(p, q)$

For any inner point p and positive integer q let $\Psi(p, q)$ be the set of triplets $(\mathscr{A}, \mathscr{E}, P)$ where $\mathscr{A}$ and $\mathscr{E}$ are trusses and P is a patch, which satisfy the following Condition **F**:

F1. $\mathscr{A}$ consists of arrows, $\mathscr{E}$ consists of edges on $\bar{V}'$.

F2. The truss $\mathscr{A} * \mathscr{E}$ is overall-even and connected.

F3. $p \in \mathrm{dom}(\mathscr{A} * \mathscr{E})$.

F4. $\mathrm{dom}(P) \subseteq \mathrm{dom}(\mathscr{A} * \mathscr{E})$.

F5. $\mathrm{sum}(P) - q \geqslant |\mathscr{E}|/r$.

Now define the family $\Phi(p, q)$ as follows: One element of this family is the special patch P_0 whose domain consists of one element p and which maps it into q. Otherwise a patch P belongs to $\Phi(p, q)$ if there are such trusses $\mathscr{A}$ and $\mathscr{E}$ that $(\mathscr{A}, \mathscr{E}, \mathscr{P}) \in \Psi(p, q)$. Based on this definition, let us prove (2). Let $|S|$ denote the cardinality of a finite set S. Estimate the cardinalities of sets

$$\Phi_j(p, q) = \{P \in \Phi(p, q) \,|\, \mathrm{sum}(P) = j\} \qquad \text{where} \quad j = 0, 1, ,2\ 3, \ldots$$

First $|\Phi_j(p, q)| = 0$ for all $j < q$, because of **F5**.

Now estimate $|\Phi_j(p, q)|$ for $j \geqslant q$. Remember that $\Phi_1(p, 1)$ contains the special element P_0 and leave it alone. Without this element, $|\Phi_j(p, q)|$ is not more than the number of different corresponding triplets $(\mathscr{A}, \mathscr{E}, P)$. If $P \in \Phi_j(p, q)$, then the corresponding trusses have (**F5**, then **F1** and **F2** allow us to apply Lemma 2)

$$|\mathscr{E}| \leqslant \mathrm{const} \cdot (j - q) \Rightarrow |\mathscr{A}| \leqslant \mathrm{const} \cdot (j - q) \Rightarrow |\mathscr{A} * \mathscr{E}| \leqslant \mathrm{const} \cdot (j - q)$$

In the case $j = q$ this implies that $\mathscr{A}$ and $\mathscr{E}$ are empty, whence and from **F4**, $\mathrm{dom}(P)$ is empty also, which is impossible since $\mathrm{sum}(P) = j \geqslant q > 0$. Thus, including P_0, $|\Phi_q(p, q)|$ equals 1 if $q = 1$ and 0 if $q > 1$.

Now let $j > q$ and prove that $|\Phi_j(p, q)|$ does not exceed an exponent in j. It follows from Euler's theorem that to every trusses $\mathscr{A}$ and $\mathscr{E}$ in question there corresponds a circuit on $\bar{V}'$, which starts and ends at p and makes $2 \cdot |\mathscr{A} * \mathscr{E}|$ steps. Every member of $\mathscr{A} * \mathscr{E}$ (all of which are edgers) is represented by two steps in the two opposite directions along the underlying edge. Let us encode $\mathscr{A} * \mathscr{E}$ by moving along this circuit step by step. At every step we must choose the edge to move along, the poles our edges has (among the $2 \cdot r$ possible ones), distribute these poles between the two ends and decide whether this edger belongs to $\mathscr{A}$ or $\mathscr{E}$. Every choice is among a bounded number of possibilities, because the degree of any vertex of $\bar{V}'$ does nor exceed $n^2 + 2n$. Now we insert into this code $P(v)$ steps of another kind when we first come to a point $v \in \mathrm{dom}(P)$ to encode values of P. The resulting code contains all the information about our triplet and its length does not exceed $2|\mathscr{A} * \mathscr{E}| + \mathrm{const} \cdot j \leqslant \mathrm{const} \cdot j$. Therefore the number of triplets in question does not exceed const^j, where

$$\sum_{P \in \Phi(p,q)} \mathrm{Prob}(P) = \sum_{j=0}^{\infty} |\Phi_j(p, q)| \cdot \varepsilon^j \leqslant \sum_{j=q+1}^{\infty} \mathrm{const}^j \cdot \varepsilon^j \leqslant (\mathrm{const} \cdot \varepsilon)^{q+1}$$

Thus obtained estimations prove (2). ▮

3. AUXILIARY GRAPHS

Through Sections 3 and 4 an inner point p, a positive integer q, and a hidden confirguration h are chosen and condition $x(p) \geqslant q$ assumed. Our purpose is to present a patch $P \in \Phi(p, q)$ which covers h. We shall present it as a result of an "inquest" which will be described in Section 4. This section makes the necessary preparations for that.

3.1. The Graph $\bar{G}$

First we describe a procedure called "segmentation" which is necessary only in $\mathbf{m} > 1$; it makes the general case "imitate" the case $\mathbf{m} = 1$. Suppose that some directed graph $\bar{G} \subseteq \bar{V}$ contains an edge $u \to v$. To segment this edge means to delete this edge from $\bar{G}$ and substitute it by a "chain"

$$u = w_0 \to w_1 \to w_2 \to \cdots \to w_{\Delta t - 1} \to w_{\Delta t} = v$$

where w_0 and $w_{\Delta t}$ are our vertices u and v, and w_k, where $k = 1, \ldots, \Delta t - 1 = t(v) - t(u) - 1$, are new intermediate vertices connected by Δt new intermediate edges. We define the time function on the newly introduced vertices by the rule $t(w_k) = t(u) + k$ for all $k = 1, \ldots, \Delta t - 1$.

Now let us call a *typed graph* a directed graph in which every vertex v has a *type*:

$$\mathrm{type}(v) \in \{A, B, S, U, X\}$$

Let us construct a typed graph $\bar{G}$ which will show us the scope of our future "inquest"; the domain of our patch will belong to the set of its vertices. Actually we shall inductively construct a sequence of typed graphs $\bar{G}_0, \bar{G}_1, \bar{G}_2,\ldots$ and at some step we stop. Some vertices of these graphs will belong to V, others will be introduced by segmentation. Along with this, we shall prove the following Condition $\mathbf{G}_i$:

G1$_i$. $x(\cdot) > 0$ for all those vertices of $\bar{G}_i$ which belong to V.

G2$_i$. A vertex of $\bar{G}_i$ has type S iff it does not belong to V.

G3$_i$. A vertex of $\bar{G}_i$ has no neighbors iff its type is U or X.

Initial Condition: Graph $\bar{G}_0$ has one vertex p of type X and no edges. Note that Conditions $\mathbf{G}_0$ hold.

Stop-Rule: As soon as no vertex of $\bar{G}_i$ has type X, we stop and define $\bar{G} = \bar{G}_i$.

Induction Step: Suppose that the graph $\bar{G}_i$ has a vertex v whose type is X. By the induction assumption, $x(v) > 0$. Consider three cases.

Case 0. Let $x(v) = h(v)$. Then v is an upstart. In this case we turn $\bar{G}_i$ into $\bar{G}_{i+1}$ just by changing type of v from X to U.

Now let $x(v) > h(v)$. Then at least one of the following two cases takes place:

Case 1. Every v-drag contains a point w where $x(w) \geqslant x(v) - h(v)$. Then, from (4), for every $k = 1,\ldots, r$ there exists $n_k(v) \in N(v)$ where

$$x(n_k(v)) \geqslant \mathbf{f}(x(N(v))) \qquad \text{and} \qquad L_k(n_k(v)) - L_k(v) \leqslant 0 \tag{8}$$

In this case we obtain $\bar{G}_{i+1}$ from $\bar{G}_i$ by changing type of the vertex v from X to A and introducing the segmented edges $n_k(v) \to v$ for all $k = 1,\ldots, r$.

Case 2. There is a point $n(v) \in N(v)$ where

$$x(n(v)) > \mathbf{f}(x(N(v))) \tag{9}$$

In this case we obtain $\bar{G}_{i+1}$ from $\bar{G}_i$ by changing type of the vertex v from X to B and introducing the segmented edge $n(v) \to v$.

All the vertices introduced by segmentation in Cases 1 and 2 do not belong to V, and are assigned the type S. The endpoints $n_k(v)$ and $n(v)$ may belong to $\bar{G}_i$, in which case they keep the same type they had in $\bar{G}_i$; otherwise they are assigned the type X.

Note that Conditions $\mathbf{G}_i$ given, Conditions $\mathbf{G}_{i+1}$ are also true in all cases.

Let us prove that G-induction stops. The number of vertices of $\bar{G}_i$ plus the number of those vertices of $\bar{G}_i$, whose type is U, increases at every step. On the other hand, all the vertices of $\bar{G}_i$ are transit $\bar{V}$-neighbors of p, whence their number is bounded.

3.2. Tree $\bar{T}$

Call vertices of $\bar{G}$ *leaves* and classify them into equivalence classes, called *branches*, by the following two rules:

- If two leaves have equal time values and a common transit $\bar{G}$-neighbor, they are equivalent.
- Only those leaves are equivalent for which it follows from the previous rule and transitivity.

Now define a directed graph $\bar{T}$ by the following two rules:

- $\mathrm{ver}(\bar{T})$ is the set of branches.
- Branches A, B given, $A \in N_{\bar{T}}(B)$ iff $\exists a \in A$, $b \in B$: $a \in N_{\bar{G}}(b)$.

Note that the leaf p is equivalent to no other leaf. So $\bar{T}$ has a vertex $\{p\}$. Graph $\bar{T}$ is a tree in which all edges are directed toward the vertex $\{p\}$. A branch B given, call $N_{\bar{G}}^{\mathrm{tran}}(B)$ its *bush* or $\mathrm{bush}(B)$. A *bush* means a bush of some branch, which is called its *root*. Say that a branch *cuts* a set S of branches if $S \subseteq N_{\bar{T}}^{\mathrm{tran}}(B)$. For any set S of branches there is just one branch that cuts S, all of whose $\bar{T}$-neighbors do not cut S; call this branch the *root* of S and denote it $\mathrm{root}(S)$. For any polar π on $\bar{G}$ its *root* or $\mathrm{root}(\pi)$ is the root of the set of branches which intersect its domain, and its *bush* is $\mathrm{bush}(\pi) = \mathrm{bush}(\mathrm{root}(\pi))$.

The fact that $\bar{T}$ is a tree assures that whenever our "inquest" (which we describe in the next section) ramifies in the space (like the "fence" on Fig. 1.1 on p. 10 in ref. 2), the resulting different paths never meet again to interfere. We need this to be able to prove by induction that the complexity of our construction will not exceed the sum of the resulting patch, multiplied by a constant.

3.3. Graphs $\bar{\nabla}(\cdot)$

For any branch A whose $\bar{T}$-neighborhood is nonempty, defined a nondirected graph $\bar{\nabla}(A)$ by the following two rules:

- $\mathrm{ver}(\bar{\nabla}(A)) = N_{\bar{T}}(A)$.

• Two different vertices B^1, B^2 of $\bar{\nabla}(A)$ are connected with an edge in $\bar{\nabla}(A)$ iff there exist

$$a \in A, \qquad b^1 \in N_{\bar{G}}^{\text{tran}}(B^1) \cap N_{\bar{G}}(a), \qquad b^2 \in N_{\bar{G}}^{\text{tran}}(B^2) \cap N_{\bar{G}}(a)$$

Lemma 3. Every graph $\bar{\nabla}(A)$ is connected.

Proof. Take any different vertices B^1 and B^2 of $\bar{\nabla}(A)$ and prove that they are connected with a path in this graph. By definitions, there are

$$a^1, a^2 \in A, \qquad b^1 \in B^1 \cap N_{\bar{G}}(a^1), \qquad b^2 \in B^2 \cap N_{\bar{G}}(a^2)$$

If $a^1 = a^2$, then B^1 and B^2 are connected with an edge. Now let $a^1 \neq a^2$. Since a^1 and a^2 belong to one branch A, they are equivalent, which means that there is such a sequence $c_0 = a_1, c_1, \ldots, c_l = a^2$ of different elements of A in which every two neighboring terms have a common proper $\bar{G}$-neighbor. This ensures that for every $i = 1, \ldots, l$ there are $e_{i-1} \in N_{\bar{G}}(c_{i-1})$ and $e'_{i-1} \in N_{\bar{G}}(c_i)$ whose transit $\bar{G}$-neighborhoods intersect. Besides that, $t(e_{i-1}) = t(e'_i)$. Therefore e_{i-1} and e'_i are equivalent; denote E_i the branch to which they belong. Now every two neighboring terms of the sequence $B^1, E_1, \ldots, E_l, B^2$ either coincide or are connected with an edge in $\bar{\nabla}(A)$. ∎

4. Q-INDUCTION

4.1. Conditions Q_i

Now we start our "inquest." For every $i = 0, 1, 2, \ldots$ we shall form a $(4 + r)$-tuple

$$\Omega_i = (\mathscr{A}_i, \mathscr{B}_i, \mathscr{C}_i, \mathscr{D}_i, P_i^1, \ldots, P_i^r)$$

where $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$, and $\mathscr{D}_i$ are trusses and P_i^j are patches. Thereby we shall also define a truss $\mathscr{Q}_i = \mathscr{A}_i * \mathscr{B}_i * \mathscr{C}_i * \mathscr{D}_i$ and the following r functions $F_i^1, \ldots, F_i^r$ on V:

$$F_i^j(v) = x(v) - P_i^j(v) \tag{10}$$

Along with constructing Ω_i, we shall prove that they satisfy the following Conditions $\mathbf{Q}_i$:

Q1$_i$. $\mathscr{A}_i$ consists of arrows, $\mathscr{B}_i$ consists of diedgers on $\bar{V}$, $\mathscr{C}_i$ consists of posiedgers on $\bar{V}'$, and $\mathscr{D}_i$ consists of negapoles.

Q2$_i$. $\mathscr{Q}_i$ is overall-even and connected.

Q3$_i$. $p \in \mathrm{dom}(\mathscr{Q}_i) \subseteq (\mathrm{ver}(\bar{G}) \cap V)$.

Q4$_i$. $\mathrm{dom}(P_i^j) \subseteq \mathrm{dom}(\mathcal{Q}_i(j))$ and $P_i^j(v) \leqslant h(v)$ for all j and all inner v.

Q5^i. $\sum_{j=1}^{r} (\mathrm{sum}(P_i^j) + \mathrm{sum}(F_i^j, \mathcal{D}_i(j))) \geqslant r \cdot x(p) + |\mathcal{B}_i| + r \cdot |\mathcal{C}_i|$.

Q6$_i$. For every j and for every $\delta \in \mathcal{D}_i$ the truss $\mathcal{Q}_i - \delta$ has only one positive j-pole and no negative j-poles in bush(δ).

Q7$_i$. For every j every j-pole of $\mathcal{Q}_i$ has a transit $\bar{G}$-neighbor, which is a j-pole of $\mathcal{D}_i$.

Initial Condition. Trusses $\mathcal{A}_{\cdot}$, $\mathcal{B}_0$, $\mathcal{C}_0$ and domains of $P_0^1, \ldots, P_0^r$ are empty and $\mathcal{D}_0$ consists of one trivial negapole, all of whose poles coincide with p. Note that Conditions $\mathbf{Q}_0$ hold.

Stop-Rule: We stop as soon as all of the following Conditions $\mathbf{S}_i$ hold:

S1$_i$. $P_i^j(v) = h(v)$ for all $v \in \mathrm{dom}(\mathcal{D}_i(j))$ for every j.

S2$_i$. Every element of $\mathrm{dom}(\mathcal{D}_i)$ is an upstart.

S3$_i$. All the members of $\mathcal{D}_i$ are trivial.

We shall also refer to the following two corollaries of Conditions $\mathbf{Q}_i$:

$$\text{no proper } \bar{G}\text{-neighbor of an element of } \mathrm{dom}(\mathcal{D}_i(j)) \text{ can belong to } \mathrm{dom}(P_i^j) \tag{11}$$

(for every j) and

$$\text{bushes of members of } \mathcal{D}_i \text{ are pairwise } \bar{G}\text{-uncomparable} \tag{12}$$

First prove (11). Assume the contrary, that is, there are

$$v = \delta^-(j) \qquad \text{and} \qquad u \in N^{\mathrm{prop}}(v) \cap \mathrm{dom}(P_i^j)$$

Then from **Q4**$_i$, $u \in \mathrm{dom}(Q_i(j))$. Then from **Q7**$_i$, u has a transit $\bar{G}$-neighbor $w \in \mathcal{D}_i(j)$. Then from Q6$_i$, $w = \delta^-(j)$, which is impossible since $w \neq v$.

Now prove (12). First note that if two bushes are $\bar{G}$-comparable, then their roots are $\bar{T}$-comparable. Now assume that there are $\delta, \delta' \in \mathcal{D}_i$ whose bushes are $\bar{G}$-comparable. Then root(δ) is a transit $\bar{G}$-neighbor of root(δ') (or vice versa, which is analogous), whence bush(δ') contains at least $2 \cdot r$ negative poles. Then, from **Q2**$_i$, it contains at least $2 \cdot r$ positive poles, which contradicts **Q6**$_i$.

4.2. Induction Step

Assume that Ω_i is constructed and Conditions $\mathbf{Q}_i$ proved. Assume also that at least one of Condition $\mathbf{S}_i$ fails and define Ω_{i+1} and prove Conditions $\mathbf{Q}_{i+1}$. (Most of these proofs are trivial, and we omit them.)

Case 1. $\mathbf{S1}_i$ fails, i.e., there are such j and $v=\delta^-(j)\in\mathrm{dom}(\mathscr{D}_i(j))$ that $h(v)>0$, but v does not belong to $\mathrm{dom}(P_i^j)$. In this case the only difference between Ω_{i+1} and Ω_i is that P_{i+1}^j differs from P_i^j at one point: $P_{i+1}^j(v)=h(v)$.

The only nontrivial proof in this case (as in most others) is that of $\mathbf{Q5}_{i+1}$. As i increases by one, the only change in $\mathbf{Q5}_{i+1}$ is in the jth term in the sum at the left side. The first addend of this term, $\mathrm{sum}(P_i^j)$, increases by $h(v)$, but the second addend, $\mathrm{sum}(F_i^j, \mathscr{D}_i(j))$, decreases by $h(v)$, so the inequality remains true.

Case 2. $\mathbf{S1}_i$ holds, but $\mathbf{S2}_i$ fails, i.e., $\mathrm{dom}(\mathscr{D}_i(j)\subseteq\mathrm{dom}(P_i^j)$ for every j, but there is some $v=\delta^-(k)\in\mathrm{dom}(D_i)$ which is not an upstart. Then, according to the Stop-Rule of G-Induction, $\mathrm{type}(v)$ is A or B. Consider these two cases.

Case 2A. $\mathrm{type}(v)=A$. Then there exists $n_k(v)$. In this case the only difference between Ω_{i+1} and Ω_i is that $\mathscr{A}_{i+1}$ results from $\mathscr{A}_i$ by introducing a k-arrow from $n_k(v)$ to v and $\mathscr{D}_{i+1}$ differs from $\mathscr{D}_i$ in only one respect: instead of δ, it contains δ_{new}, which has only one pole different from that of δ, namely $\delta_{\mathrm{new}}^-(k)=n_k(v)$.

All we need to prove in this case is $\mathbf{Q5}_{i+1}$ again. All that changes is the kth term at the left side. Let us prove that it does not decrease, which comes to $F_{i+1}^k(n_k(v))\geqslant F_i^k(v)$, which can be rewritten as

$$F_i^k(n_k(v))\geqslant F_i^k(v) \tag{13}$$

Let us prove it. Since $\mathbf{S1}_i$ holds, $F_i^k(v)=x(v)-h(v)=\mathbf{f}(x(N(v)))$. On the other hand, from (11), $n_k(v)$ cannot belong to $\mathrm{dom}(P_i^k)$, when $F_i^k(n_k(v))=x(n_k(v))$, and (13) follows from (8).

Case 2B. $\mathrm{type}(v)=B$. Then there exists $n(v)$. In this case the only difference between Ω_{i+1} and Ω_i is that $\mathscr{B}_{i+1}$ results from $\mathscr{B}_i$ by introducing a dipole from $n(v)$ to v and $\mathscr{D}_{i+1}$ differs from $\mathscr{D}_i$ in only one respect: instead of δ, it contains δ_{new}, which has only one pole different from that of δ, namely $\delta_{\mathrm{new}}^-(k)=n(v)$.

All we need to prove in this case is $\mathbf{Q5}_{i+1}$ again. Now the right side increases by 1, and we must prove that the left side increases also, which comes to $F_i^k(n(v))>F_i^k(v)$. Here $F_i^k=x(v)-h(v)=\mathbf{f}(x(N(v)))$ from $\mathbf{S1}_i$, while $F_i^k(n(v))=x(n(v))$ from (11) and $\mathbf{Q4}_i$. Thus the inequality to prove follows from (9).

Case 3. $\mathbf{S1}_i$ and $\mathbf{S2}_i$ hold, but $\mathbf{S3}_i$ fails. So $\mathscr{D}_i$ contains a nontrivial negapole δ. Let us prove that $\mathrm{dom}(\delta)$ and $\mathrm{root}(\delta)$ do not intersect. Assume the contrary: there is some $\delta^-(k)\in\mathrm{root}(\delta)$. Since all the elements of dom(δ) are upstarts, they have no $\bar{G}$-neighbors. (From **G3**$_i$ a vertex of $\bar{G}$ has no $\bar{G}$-neighbors iff its type is U, that is, iff it is an upstart.) Since $\delta^-(k)$ has no $\bar{G}$-neighbors, root(δ) $= \{\delta^-(k)\}$. Then δ is trivial, which contradicts our assumption.

Thus, every pole of δ belongs to the transit $\bar{G}$-neighborhood of some $\bar{T}$-neighbor of root(δ). This allows us to define a posipole π on $N_{\bar{T}}(\text{root}(\delta))$ by the following rule: for every k its kth pole $\pi^+(k)$ is that $\bar{T}$-neighbor of root(δ), whose bush contains $\delta^-(k)$. From Lemma 3 the graph $\bar{\nabla}(\text{root}(\delta))$ is connected. Therefore there exists a spanning kit of π on $\bar{\nabla}(\text{root}(\delta))$. For every member ε of its spanning truss we form a posiedger $\gamma(\varepsilon)$ on $\bar{V}'$ as follows. Remember that ε is a posiedger, whence its domain has two elements, say A and B, which are vertices of the graph $\bar{\nabla}(\text{root}(\delta))$, connected with an edge. So there are leaves

$$c \in V, \qquad a \in N_G^{\text{tran}}(A) \cap N_{\bar{G}}(c), \qquad b \in N_G^{\text{tran}}(B) \cap N_{\bar{G}}(c)$$

Now define our posiedger γ as follows:

$$\forall k: \quad \gamma^+(k) = \begin{cases} a & \text{if } \varepsilon(k) = A \\ b & \text{if } \varepsilon(k) = B \end{cases}$$

Truss Γ_i consists of these γ:

$$\Gamma_i = \langle \gamma(\varepsilon) \colon \varepsilon \in \text{span-truss}(\pi, \bar{\nabla}(\text{root}(\delta))) \rangle$$

In Case 3 the only difference between Ω_{i+1} and Ω_i is that $\mathscr{C}_{i+1} = \mathscr{C}_i * \Gamma_i$ and $\mathscr{D}_{i+1}$ differs from $\mathscr{D}_i$ as follows. Denote

$$\mathscr{T}_{i+1} = A^{i+1} * \mathscr{B}_{i+1} * \mathscr{C}_{i+1} \quad \text{and} \quad V_{\text{span}} = \text{ver}(\text{span-tree}(\pi, \bar{\nabla}(\text{root}(\delta))))$$

From (6), $\mathscr{T}_{i+1}$ is 1-even on every $B \in V_{\text{span}}$. Thus, for every j every branch $B \in V_{\text{span}}$ serves as the jth pole just for one member of $\langle \pi \rangle *$ span-truss($\pi, \bar{\nabla}(\text{root}(\delta))$). This means that for every $B \in V_{\text{span}}$ and for every j the truss $\langle \delta \rangle * \Gamma_i$ has just one positive jth pole in bush(B), and let it be the negative jth pole of the new negapole δ_B that corresponds to B. Now define

$$\mathscr{D}_{i+1} = (\mathscr{D}_i - \delta) * \langle \delta_B \colon B \in V_{\text{span}} \rangle$$

All we need to prove in this case are **Q2**$_{i+1}$ and **Q5**$_{i+1}$. Condition **Q2**$_{i+1}$ follows from (7) and (6). Let us prove **Q5**$_{i+1}$. As time grows its right side grows by $r \cdot |\Gamma_i|$ and we must prove that the left side grows at least as much. Since $P_{i+1}^j = P_i^j$ for all j, the left side grows by

$$\text{sum}(F_i, \mathscr{D}_{i+1}) - \text{sum}(F_i, \mathscr{D}_i) = \text{sum}(F_i, \langle \delta_B \colon B \in V_{\text{span}} \rangle) - \text{sum}(F_i, \delta) \quad (14)$$

Note that for every j the j-pole of δ coincides with some j-pole of $\langle \delta_B$: $B \in V_{\text{span}} \rangle$), and the corresponding terms of our sums cancel out. Let us call all the other poles of $\langle \delta_B$: $B \in V_{\text{span}} \rangle$) intermediate. Let us prove that no intermediate j-pole belongs to $\text{dom}(P_i^j)$. Assume that there is an intermediate j-pole $v \in \text{dom}(P_i^j)$. Then from **Q4**$_i$, $v \in \text{dom}(\mathscr{Q}_i(j))$, that is, v is a j-pole of $\mathscr{Q}_i$. Then from **Q7**$_i$, v has a transit $\bar{G}$-neighbor $w \in \mathscr{D}_i(j)$. Since $w \in \text{bush}(\delta)$, (12) ensures that w cannot belong to the bush of any other member of $\mathscr{D}_i$. Thus w is the j-pole of δ. Thus $\delta^-(j)$ is a transit $\bar{G}$-neighbor of v. But $\delta^-(j)$ coincides with some nonintermediate j-pole of our spanning truss. Thus two j-poles of some different members of our spanning truss are $\bar{G}$-comparable, which is impossible, because they belong to bushes of different $\bar{T}$-neighbors of $\text{root}(\delta)$. Thus no intermediate j-pole belongs to $\text{dom}(P_i^j)$. Thus the value of F_i^j at any intermediate j-pole v equals $x(v)$, which is positive, because v is a vertex of $\bar{G}$ (see **G1**$_i$). Thus the difference (14) is not less than the number $t(|V_{\text{span}}| - 1)$ of the intermediate poles. From (5) this number equals $r|\Gamma_i|$.

Thus Ω_{i+1} is defined and Conditions $\mathbf{Q}_{i+1}$ proved in all cases.

4.3. Result of Q-Induction

Let us prove that Q-induction stops. From **Q4**$_i$ the sum $\text{sum}(P_i^1) + \cdots + \text{sum}(P_i^r)$ is bounded, whence Case 1 cannot occur infinitely. From **Q3**$_i$ and (12) $|\mathscr{D}_i|$ cannot be greater than $|\text{ver}(\bar{G})|$, whence Case 3 also cannot occurs infinitely, because its every occurence increases $|\mathscr{D}_i|$. Now $\text{sum}(F_i, \mathscr{D}_i)$ is also bounded, because every value of functions F_i^j on vertices of $\bar{G}$ cannot exceed $t(p)$. Thus, the left side of **Q5**$_i$ is bounded, whence the right side is also bounded, whence $|\mathscr{B}_i|$ and $|\mathscr{C}_i|$ are bounded, whence Case 2 also cannot occur infinitely.

When Q-Induction stops, we obtain the trusses $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$, $\mathscr{D}_i$ and the patches $P_i^1, \ldots, P_i^r$ for which Conditions $\mathbf{Q}_i$ and $\mathbf{S}_i$ hold, and define

$$\mathscr{A} = \mathscr{A}_i, \qquad \mathscr{E} = \mathscr{B}_i * \mathscr{C}_i, \qquad P = P_i^1 \cap \cdots \cap P_i^r \tag{15}$$

Remember that our goal was to present a patch P; now it is given by (15). The inclusion $h \in P$ is assured by **Q4**$_i$. It remains to prove that $P \in \Phi(p, q)$.

First let p be an upstart. In this case G-induction stops without making a step. Q-induction makes r steps, at every one of which Case 1 holds. When it stops, $\mathscr{A}_r$, $\mathscr{B}_r$, and $\mathscr{C}_r$ are empty, $\mathscr{D}_r$ consists of one trivial negapole all of whose poles coincide with p, and $\text{dom}(P_r^j) = \{p\}$ for all j, where $P = P_0$.

Now let p not be an upstart. Let us prove that the triplet $\mathscr{A}$, $\mathscr{E}$, P

defined by (15) satisfies Conditions **F**. In this case G-induction makes at least one step, $\bar{G}$ contains more than one vertex, and $\mathscr{A} * \mathscr{E}$ is nonempty. $\mathscr{A} * \mathscr{E}$ is connected and overall-even, because $\mathscr{Q}_i$ is, and because the only difference between them is $\mathscr{D}_i$, which consists of trivial polars from **S3**$_i$. For the same reason $\mathrm{dom}(\mathscr{A} * \mathscr{E}) = \mathrm{dom}(\mathscr{Q}_i)$. Then:

F1 follows from **Q1**$_i$.

F2 follows from **Q2**$_i$ and **S3**$_i$.

F3 follows from **Q3**$_i$.

F4 follows from **Q4**$_i$.

F5 follows from **Q4**$_i$, **Q5**$_i$, from the fact that P belongs to all P_i^k, and from the fact that when Q-Induction stops, $\mathrm{sum}(F_i, \mathscr{D}_i) = 0$. [This follows from (10) and **S2**$_i$.]

Thus Proposition 1 is proved. ∎

Let us see that our constructions give for Example 1 of Part I when $p = (0, T)$, $q = 1$, and the hidden variables equal 0 everywhere except the points $(-T, 0)$, $(-T+1, 0)$,..., $(-1, 0)$, $(0, 0)$, where they equal 1. These points are the upstarts which ultimately cause the event $x(p) \geqslant q$. Points of the triangle

$$\{(s, t) \text{ where } -T + t \leqslant s \leqslant 0 \text{ for all } t = 0,..., T\}$$

are vertices of $\bar{G}$. Every upstart is a branch of its own (which is always true) and all of these branches are $\bar{T}$-neighbors of the branch $\{(-T+1, 1),..., (0, 1)\}$, which is the only $\bar{T}$-neighbor of the branch $\{(-T+2, 2),..., (0, 2)\}$, which is the only $\bar{T}$-neighbor of the branch $\{(-T+3, 3),..., 3,),..., (0, 3)\}$, and so on to the branch $\{(0, T)\}$. When Q-Induction stops, the truss $\mathscr{A}_i$ consists of two "chains" of arrows

$$(-T, 0) \to (-T+1, 1) \to \cdots \to (-1, T-1) \to (0, T)$$

and

$$(0, 0) \to (0, 1) \to \cdots \to (0, T-1) \to (0, T)$$

The truss $\mathscr{B}_i$ is empty. ($\mathscr{B}_i$ is always empty in Examples 1–3.) The truss $\mathscr{C}_i$ consists of T posipoles, which also form a "chain"

$$(-T, 0) - (-T+1, 0) - \cdots - (-1, 0) - (0, 0)$$

One can see that the three "chains" formed by $\mathscr{A}_i$ and $\mathscr{C}_i$ surround our triangle. Thus the resulting truss $\mathscr{T}_i$ has the shape of a contour. This takes place in Example 1 always, because $r=2$ in this example. Whenever $r \geqslant 3$, the resulting truss does not have the form of just a contour; its geometrical shape ramifies.

5. PROOF OF PROPOSITION 2

Proof of Proposition 2 is similar to and based on the proof of Proposition 1. In this case for every p and q we construct two finite families $\Phi'(p, q)$ and $\Phi''(p, q)$ of patches which cover the event $(x(p) \geqslant q)$ and satisfy the following conditions where $C = \text{const}$:

$$\sum_{P \in \Phi'(p,q)} \text{Prob}(P) \leqslant \begin{cases} \varepsilon + (C\varepsilon)^2 & \text{if} \quad q = 1 \\ (C\varepsilon)^{q+1} & \text{if} \quad q > 1 \end{cases} \tag{16}$$

$$\sum_{P \in \Phi''(p,q)} \text{Prob}(P) \leqslant \text{const} \cdot t(p) \cdot (C \cdot \varepsilon)^{q+1+\text{const} \cdot M} \tag{17}$$

This ensures Proposition 2, because any constant T given, while $t(p) < T^M$, the right part of (17) is still less than $\text{const} \cdot (\text{const} \cdot \varepsilon)^{q+1+\text{const} \cdot M}$, whence (1) follows. ▮

5.1. Description of $\Phi'(p, q)$ and $\Phi''(p, q)$

Remember the definition of Δ in Part I and denotes $\text{diam}_\Delta(S) = \Delta + \text{diam}(S)$. Let us first prove that for any $v \in V$

$$\text{diam}_\Delta(N_{\bar{G}}^{\text{tran}}(v)) \leqslant \Delta + \sum_{w \in N_{\bar{G}}(v)} \text{diam}_\Delta(N_{\bar{G}}^{\text{tran}}(w)) \tag{18}$$

Call two point-sets *Δ-close* if the distance between them does not exceed Δ. (Note that any nonempty set is Δ-close to itself.) For any family F of sets let union(F) be the union of its elements. Call a family of point-sets *Δ-connected* if it is possible to go from any element of its union to any other, every step made between elements of Δ-close sets. If a family F of point-sets is Δ-connected, then

$$\text{diam}_\Delta(\text{union}(F)) \leqslant \sum_{S \in F} \text{diam}_\Delta(S) \tag{19}$$

which can be proved by induction. Now (18) follows from (19) applied to the case when the family consists of the point v and transit $\bar{G}$-neighborhoods of all $\bar{G}$-neighbors of v. (Here we assume that the transit $\bar{G}$-neighborhood of any point is the union of this point and transit $\bar{G}$-neighborhoods of its $\bar{G}$-neighbors. To assure this we must have made all the arbitrary choices in the course of G-Induction in some standard way.)

Denote by $\mathscr{P}_M$ the process on the finite volume $V_M = \mathbf{Z}_M^d \cdot \text{Time}$. Call its elements M-points, while ∞-points will be the elements of $V_\infty = \mathbf{Z}_\infty^d \cdot \text{Time}$. Define a map fin: $V_\infty \mapsto V_M$ as follows: $\text{fin}(s, t) = (s', t)$, where components of s' are residues of components of s modulo M. The opposite map inf: $V_M \mapsto V_\infty$ is quite simple: $\text{inf}(s, t) = (s, t)$. Call elements of the finite and infinite hidden space *hidden M-configurations* and *∞-configurations*. To every hidden M-onfiguration h there corresponds a periodical hidden *∞-configuration* h' defined by the rule $h'(v) = h(\text{fin}(v))$ for all $v \in V_\infty$. We shall use the auxiliary infinite process $\mathscr{P}'_\infty$ on V_∞ induced by the finite hidden measure with this map. Note that the restriction of the process $\mathscr{P}'_\infty$ to any point v coincides with the restriction of $\mathscr{P}_M$ to $\text{fin}(v)$.

Accordingly, we consider patches of two kinds: ∞-patches and M-patches (with correspond to the infinite and finite systems). An ∞-patch P_∞ given, $P_M = \text{fin}(P_\infty)$ is the M-patch which maps any v into the maximum of those numbers into which P_∞ maps preimages of v.

Say that a set $S \subset V_\infty$ *overlaps* if there are such different $v, v' \in S$ that $\text{fin}(v) = \text{fin}(v')$.

To define $\Phi'(v, k)$: One element of it is the M-patch P_0 whose domain consists of one element v and which maps it into $h(v)$ (just as in the infinite case). Otherwise, an M-patch P_M belongs to $\Phi'(v, k)$ if it is representable as $\text{fin}(P_\infty)$, where P_∞ is such an ∞-patch for which there are such trusses $\mathscr{A}$ and $\mathscr{E}$ on V_∞ that $(\mathscr{A}, \mathscr{E}, P_\infty) \in \Psi(\text{inf}(v), k)$ and $\text{dom}(P_\infty)$ does not overlap. Family $\Phi'(p, q)$ satisfies (16), which can be proved in just the way (2) was.

To define $\Phi''(p, q)$: An M-patch P_M belongs to $\Phi''(p, q)$ if it is representable as $\text{fin}(P_\infty)$, where P_∞ is such an ∞-patch for which there are such trusses $\mathscr{A}$ and $\mathscr{E}$ on V_∞ and such an M-point $\text{vice}(p) \in N_V^{\text{tran}}(p)$ that $(\mathscr{A}, \mathscr{E}, P_\infty) \in \Psi(\text{vice}(p), q)$ and $\text{dom}(P_\infty)$ does not overlap and

$$\text{diam}_\Delta(\text{dom}(\mathscr{A} * \mathscr{E})) \geqslant \frac{M - \Delta}{n} \tag{20}$$

Let us prove that the family $\Phi''(p, q)$ satisfies (17). From (20) and the definition of $\text{diam}_\Delta(\cdot)$

$$\text{diam}(\text{dom}(\mathscr{A} * \mathscr{E})) \geqslant \frac{M - \Delta}{n} - \Delta$$

Note that if a connected truss $\mathscr{T}$ consists of edges on $\bar{V}'$, then $\text{diam}(\text{dom}(\mathscr{T})) \leqslant \text{const} \cdot |\mathscr{T}|$. Then $|\mathscr{A} * \mathscr{E}| \geqslant \text{const} \cdot M - \text{const}$ from **F2**. Now, due to **F1** and **F2**, we can apply Lemma 2, which gives $|\mathscr{E}| \geqslant \text{const} \cdot M - \text{const}$. Note that if $\text{dom}(P_\infty)$ does not overlap, then

$\mathrm{sum}(\mathrm{fin}(P_\infty)) = \mathrm{sum}(P_\infty)$. Then, from **F5** and the fact that $\mathrm{dom}(P_\infty)$ does not overlap,

$$\mathrm{sum}(P_M) = \mathrm{sum}(P_\infty) \geqslant C_1 \cdot M - C_2$$

where $C_1 > 0$ and C_2 are constants. Now let us prove (17) by evaluating the cardinality of the sets

$$\Phi_j''(p, q) = \{S \in \Phi''(p, q) \mid |S| = j\}$$

As we have proved, $|\Phi_j''(p, q)| = 0$ for all $j < C_1 \cdot M - C_2$.

Now let $j \geqslant C_1 \cdot M - C_2$. From **F5**, $|\mathscr{E}| \leqslant r \cdot (j - q)$, then from Lemma 2, $|\mathscr{A} * \mathscr{E}| \leqslant \mathrm{const} \cdot (j - q)$. Thus $|\Phi_j''(p, q)|$ is 0 for $j < q$ and does not exceed $\mathrm{const}^{(j-q)}$ for $j \geqslant q$, which can be proved based on Euler's theorem as in the infinite case. Hence for small enough ε

$$\sum_{P \in \Phi''(p,q)} \varepsilon^{\mathrm{sum}(P)} \leqslant \mathrm{const} \cdot M^d \cdot t(p) \sum_{j = \max\{q, C_1 \cdot M - C_2\}}^{\infty} (\mathrm{const} \cdot \varepsilon)^j$$

which is less than the right part of (17) with suitable constants.

5.2. The Patch

Given an M-point p, a number q, and a hidden M-configuration h such that $x(p) \geqslant q > 0$, we apply constructions of the infinite case to the ∞-point $\mathrm{inf}(p)$ and the hidden ∞-configuration h' to obtain the triplet $(\mathscr{A}, \mathscr{E}, P)$ defined by (15). Let $\mathscr{T}_\infty(p, h, q) = \mathscr{A} * \mathscr{E}$ be the concatenation of the first two terms of this triplet, and $P_\infty(p, h, q)$ be the last term of this triplet. Let us prove that if $\mathrm{dom}(P_\infty(p, h, q))$ overlaps, then p has a transit $\bar{G}$-neighbor $\mathrm{vice}(p)$ for which $\mathrm{dom}(P_\infty(\mathrm{vice}(p), h, q))$ does not overlap, and

$$\mathrm{diam}_\Delta(\mathrm{dom}(\mathscr{T}(\mathrm{vice}(p), h, q))) \geqslant \frac{M - \Delta}{n} \tag{21}$$

Take the set of those points $w \in N_{\bar{G}}^{\mathrm{tran}}(p)$ for which $\mathrm{dom}(P_\infty(w, h, q))$ overlaps, and take a point z in this set whose time is the smallest. Let us prove by contradiction that at least one $\bar{G}$-neighbor of z fits our need. Assume the contrary.

$$\forall w \in N_{\bar{G}}(z){:}\quad \mathrm{diam}_\Delta(\mathrm{dom}(\mathscr{T}(w, h, q))) < \frac{M - \Delta}{n}$$

Then from **F5** and (18)

$$\mathrm{diam}(\mathrm{dom}(P_\infty(z, h, q))) < \mathrm{diam}_\Delta(\mathrm{dom}(P_\infty(z, h, q))) \leqslant \mathrm{diam}_\Delta(N_{\bar{G}}^{\mathrm{tran}}(z)) < M$$

But this is impossible, because if some $S \subset V_\infty$ overlaps, then $\mathrm{diam}(S) \geqslant M$.

Now we can define a patch $P_M \in \Phi'(p, q) \cup \Phi''(p, q)$ for any M-point p, any hidden M-configuration h, and any $q \leqslant x(p)$ as follows:

$$P_M = \begin{cases} \mathrm{fin}(P_\infty(p, h, q)) & \text{if } \mathrm{dom}(P_\infty(p, h, q)) \text{ does not overlap} \\ \mathrm{fin}(P_\infty(\mathrm{vice}(p), h, q)) & \text{otherwise} \end{cases} \tag{22}$$

Every patch defined by the upper line of (22) belongs to $\Phi'(p, q)$, and this can be proved as in the infinite case. To prove that every patch defined by the lower line of (22) belongs to $\Phi''(p, q)$, note that (20) follows from (21) and all the other assertions can be proved as in the infinite case. Thus Proposition 2 is proved. ▌

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